Elastic Snap-Through Analysis of Curved Plates Using Discrete Elements

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Theme

THIS paper describes a piecewise linear incremental discrete-element procedure for the analysis of geometrically nonlinear snap-through buckling responses of elastic plates with initial deflections subjected to lateral loads. Based on the variation of curvature along the load-displacement curve, the unequal load increments and decrements are chosen. In order to proceed past the points of maximum and minimum where the stiffness matrices are singular, a small increment of displacements is applied for one step. Formulations and procedures are demonstrated by the use of a rectangular plate element with initial curvature. Static symmetrical snap-through buckling responses are obtained for some typical plates.

Content

The present development is based on the von Kármán's large deflection assumptions for plate, in which the nonlinear strain-displacement equations are

$$\varepsilon_x = u_{,x} - zw_{,xx} + \tfrac{1}{2}(w + w_0)_{,x}^2 - \tfrac{1}{2}(w_0)_{,x}^2 \qquad (1a)$$

$$\varepsilon_y = v_{,y} - zw_{,yy} + \tfrac{1}{2}(w + w_0)_{,y}^2 - \tfrac{1}{2}(w_0)_{,y}^2 \qquad (1b)$$

$$\gamma = u_{,y} + v_{,x} - 2z(w_{,xy})^2 + (w + w_0)_{,x}(w + w_0)_{,y} - w_{0,x}w_{0,y} \qquad (1c)$$

where u, v, and w are displacement components and the subscript "0" indicates initial quantity. Eqs. (1) are valid when the slopes of the deformed plate are small relative to the undeformed geometry.

The strain energy for a deformed plate element is obtained by substituting Eqs. (1) into the standard integration expression for energy written in terms of products of strains and other constants. The total potential energy is the sum of the strain energy and the work due to external loads.

For any aprticular discrete element, the stiffness equations are obtained by substituting the displacement functions into the potential energy expression, integrating, and then differentiating with respect to each nodal degree of freedom in turn. The contributions of the individual elements are added to provide the stiffness equations for the whole structure. It is of the following form:

$$\{P\} = \{[K] + [N_0] + [N_1] + [N_2]\}\{\Delta + \Delta_0\} \qquad (2)$$

where $\{P\}$ and $\{\Delta\}$ are vectors of nodal loads and displacements, respectively, $[K]$ is the linear stiffness matrix; $[N_0]$, $[N_1]$, and $[N_2]$ are the incremental stiffness matrices containing the zero-, first-, and second-order gross displacements $\{\Delta + \Delta_0\}$, respectively. Eq. (2) is readily applicable for the direct iterative analysis.

Performing first-order Taylor expansion with respect to the equilibrium equation [Eq. (2)], a linearized incremental stiffness formulation is obtained

$$\{\delta P\} = \{[K] + [N_0] + 2[N_1] + 3[N_2]\}\{\delta\Delta\} \qquad (3)$$

where δ is an incremental operator. Eq. (3) may be solved in an inversed form where $\{\delta\Delta\}$ is obtained in terms of load increment $\{\delta P\}$.

Employing the stress-strain relationships and statistically equilibrium conditions, the nodal membrane forces $\{P_u\}$ and $\{P_v\}$ are obtained in terms of the nodal membrane displacements $\{u\}$ and $\{v\}$ and the quadratic functions of nodal flexural displacements $\{w + w_0\}$

$$\begin{Bmatrix} P_u \\ \overline{P_v} \end{Bmatrix} = [K_{uv}]\begin{Bmatrix} u \\ \overline{v} \end{Bmatrix} + \{w + w_0\}^T[S]\{w + w_0\} \qquad (4)$$

where $[K_{uv}]$ is the membrane portion of $[K]$; and $[S]$ is a constant matrix.

Once the flexural displacements $\{w + w_0\}$ are obtained from Eq. (3) and the membrane edge conditions are incorporated, Eq. (4) becomes a plane-stress problem. Equation (4) furnishes the distribution of membrane stress and displacement.[1]

In the following incremental procedure, the load-increments are controlled by a geometric series with

$$\text{load level at step } N = \sum_{i=1}^{N} cr^{i-1} \qquad (5)$$

where c and r are constants. The distribution of load-increment is ascendent or descendent when r is greater or smaller than unity. A detailed discussion on how to choose c and r on the basis of the curvature variation is given in Ref. 4.

When applying the incremental procedure, difficulty arises when the maximum (or minimum) point is approached. At such peak point, the total stiffness matrix is singular and the slope of the load-displacement curve becomes infinite. It is impossible to apply a load increment there no matter how small its value is chosen. In order to proceed past this point, it is suggested to apply a small increment of displacement instead of load for one step. The distribution of displacement-increment is assumed to have the same pattern as that obtained at the end of the previous step. Although error is introduced by applying the displacement-increment near the peak, it can be reduced to any desired level by refining the sizes of increment when the peak is approached.

Two numerical examples are illustrated. The first one is a long rectangular plate with edges simply supported with no membrane displacement. The plate is initially curved with the function

$$\bar{w}_0 = W_0 \sin(\pi y/a) \qquad (6)$$

where W_0 is the maximum initial deflection; and a is the plate width. The plate is under uniform pressure p acting in the direction opposite to the initial deflection. Six conforming rectangular plate bending elements are used to idealize half of a typical strip across the plate width. The slopes at half plate width are set to zero so that only symmetrical buckling mode is obtained. The results for dimensionless load vs net maximum deflection for different initial curvatures ($w_0/h = -0.5, -1, -2$) are plotted in Fig. 1 where E is the Young's modulus and h is the plate thickness. The small circles provide the indication of the load-increments chosen. For the case that $W_0/h = -0.5$, the load-increments are

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ascendent and are controlled by Eq. (5) with $c = 2.0$ and $r = 1.2$. For the case that $W_0/h = -1$, a combination of ascendent, descendent, and equal load-increments is used. From origin to A, four load-increments with $c = 4.0$ and $r = 0.75$ are applied. From A to B, a vector of displacement-increments with its maximum value equal to $0.15h$ is applied. From B to C, five equal load-decrements with $c = 1.5$ and $r = 0$ are applied. From C to D, a vector of displacement-increments with its maximum value equal to $0.15h$ is applied. Finally, load increments with $c = 2.0$ and $r = 1.2$ are applied from D on. For the case $W_0/h = -2.0$, the nonlinear behavior is very pronounced. From the origin to B, the 15 load-increments are descendent with $c = 9.5$ and $r = 0.9$. At B, a vector of displacement increments with its maximum value equal to $0.12h$ is applied. From C to F, 11 steps of ascendent load decrements with $c = 1.5$ and $r = 1.2$ are applied. From F to I, 10 steps of descendent load decrements with $c = 6.5$ and $r = 0.85$ are applied. From I to J, a vector of displacement increments with its maximum value equal to $0.12h$ is applied. From J on, the load increments are ascendent with $c = 2.0$ and $r = 1.2$.

A basis for comparison and evaluation of the present results may be found in Ref. 2. Vol'mir uses a single sinusoidal function to represent the deflection shape and obtains a solution by minimizing the potential energy. His results are also shown in Fig. 1. The agreement between the two different solutions is, in general, close. However, the discrepancy for the curve with $W_0/h = -2$ is seen to be quite pronounced in the range of minimum load. Quite obviously, a single sinusoidal deflection is too approximate to represent a snap-buckling plate that has multiple waveforms. The multiple waveforms are, however, obtained by the present discrete-element solutions and shown in the original report.

To illustrate the versatility of the present finite element method, the writer next considers an example of a square plate with all edges clamped with no rotation and in-plane movement. Only a quadrant with 16 elements is under analysis. Along the two center lines of the plate the normal slopes are set to zeros so that only symmetrical buckling mode is obtained. The plate is assumed to be initially curved with the following deflection function

$$\bar{w}_0 = W_0 \sin^2(\pi x/a)\sin^2(\pi y/a) \tag{7}$$

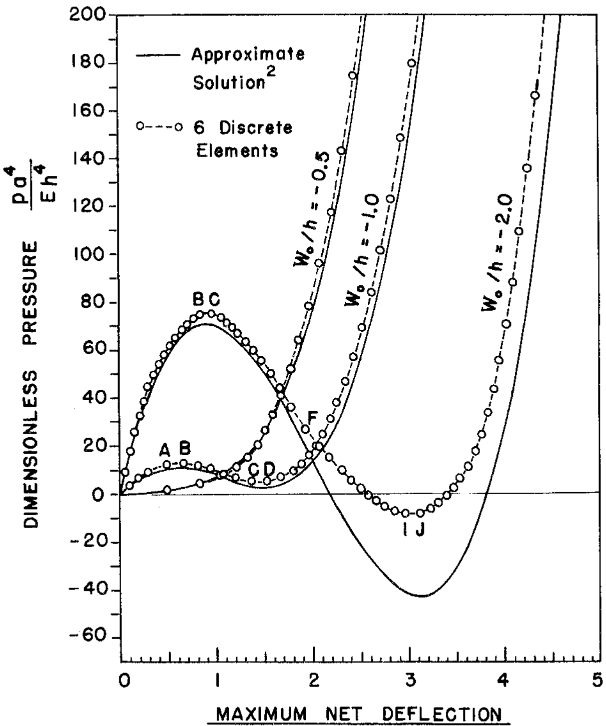

Fig. 1 Symmetrical buckling of a simply supported long plate with initial deflection in opposite direction to the load.

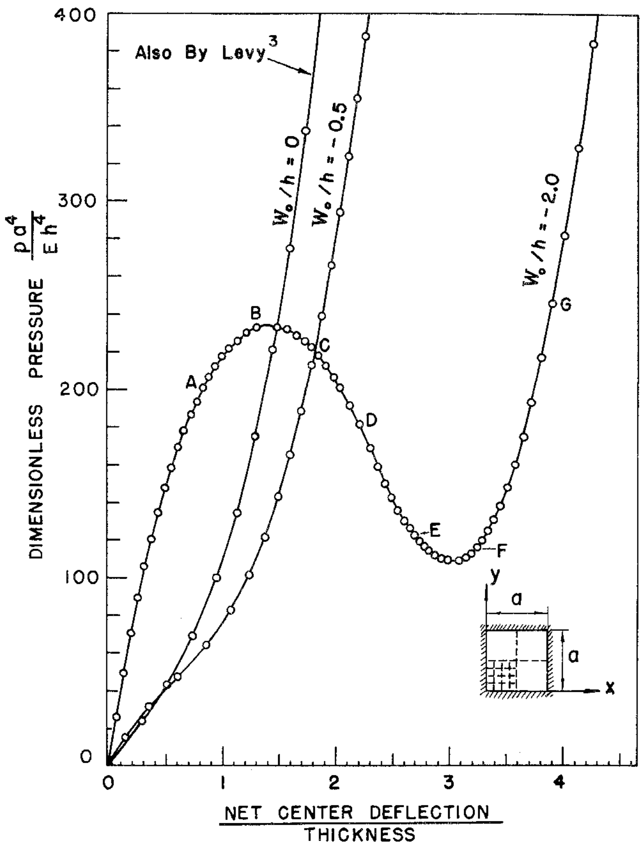

Fig. 2 Symmetrical buckling of a clamped square plate with initial deflection in opposite direction to the load.

The results for the dimensionless pressure vs net center deflection for different initial curvatures ($W_0/h = 0$, -0.5, and -2.0) are shown in Fig. 2. The sizes for all the load increments are shown by small circles. For the case that $W_0/h = 0$, the load increments are ascendent. An alternative Fourier series solution due to Levy[3] is also shown in Fig. 2 for comparison and total agreement is found. For the case that $W_0/h = -0.5$, the load increments are also ascendent. For the case that $W_0/h = -2.0$, a combination of ascendent and descendent load increments is used. The buckling waveforms at different loading stage are given in the original report.

It is noted that the present piecewise linear incremental method may be inferior to some sophisticated iterative methods such as modified Newton-Raphson's method.[5] It is also noted that in the foregoing examples, the analysis did not include the possibility of an asymmetrical bifurcation. However, the asymmetric case can straightforwardly be treated by releasing the zero-slope constraints along the center lines of the plates.

This development may provide a practical procedure for the application of discrete-element technique to the prediction of snap-buckling responses of imperfect plate and shell systems.

References

[1] Yang, T. Y., "A Finite Element Procedure for Large Deflection Analysis of Plates with Initial Deflections," *AIAA Journal*, Vol. 9, No. 8, Aug. 1971, pp. 1468–1473.

[2] Vol'mir, A. S., *Flexible Plates and Shells*, Air Force Flight Dynamics Lab., TR-66-216, 1967, Wright-Patterson Air Force Base, Ohio, p. 84.

[3] Levy, S., "Square Plate with Clamped Edges Under Normal Pressure Producing Large Deflections," TR 740, 1942, NACA.

[4] Yang, T. Y., "Finite-Displacement Plate Flexure by the Use of Matrix Incremental Approach," *International Journal for Numerical Methods in Engineering*, Vol. 4, No. 1, Jan.–Mar., 1972.

[5] Haisler, W. E., Stricklin, J. A., and Stebbins, F. J., "Development and Evaluation of Solution Procedures for Geometrically Nonlinear Structural Analysis by the Direct Stiffness Method," AIAA Paper 71–356, Los Angeles, Calif., 1971.